Systems of Differential Equations

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Seminar in Engineering Analysis

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California State University Northridge

Outline

- Review last class
- Midterm Exam October 18 covers material up to and including homework due today
- Origin of systems of differential equations
- Solving systems of equations
  - Combining into one equation
  - Matrix approach

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Review Last Class

- Analysis of forced vibrations as example of nonhomogenous equations
- Examine higher order differential equations
  - Focus on constant coefficient equations
  - Similar approach to second order except that n^{th} order equation required for λ_k
- Use undetermined coefficients for nonhomogneous solutions

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Review Forced Vibrations

- Imposed force gives ODE $md^2y/dt^2 + cdy/dt + ky = f(t)$ ($\omega_0^2 = k/m - c^2/4m^2$)
- $y_H = e^{-ct/2m}(C \sin \omega_0 t + D \cos \omega_0 t)$
- Consider example where $f(t) = F_0 \cos \omega t$
- $y = y_H + y_P = y_H + A \sin \omega t + B \cos \omega t$

$$A = \frac{F_0 \omega c}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2} \qquad B = \frac{mF_0\left(\omega_0^2 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}$$

$$C = \frac{v_0}{\omega_0} + \frac{c(y_0 - B)}{2m\omega_0} - \frac{A\omega}{\omega_0} \qquad D = y_0 - B$$

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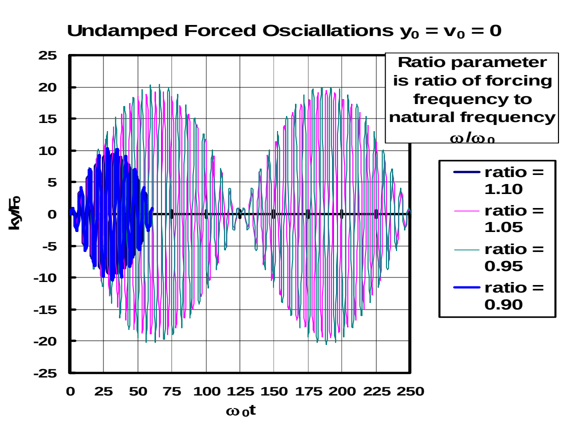

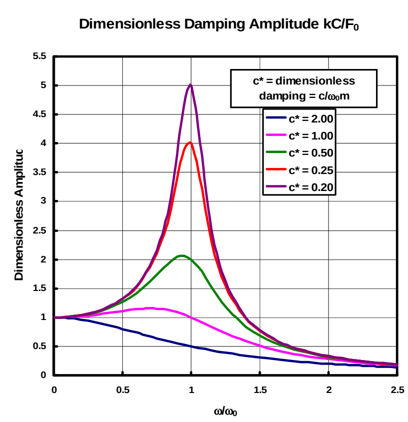

- As $c^2/m^2\omega_0^2$ becomes smaller, location of ω/ω_0 for maximum becomes one and maximum amplitude becomes $m\omega_0/c$

Higher Order Equations

- n^{th} order ODE with constant coefficients
- Solution is $y = y_H + y_P$

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = r(x)$$

- Homogenous solution $y_H = \sum_{k=1}^{n} C_k e^{\lambda_k x}$
- λ_k are solutions to the equation $\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0$
- Multiple and complex roots

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Higher Order Equations II

- y_P may be found by undetermined coefficients or variation of parameters
  - Use same process for method of undetermined coefficients
  - May have e^{ax}, sin ax, or cosine ax in r(x) where a root of homogenous solution
    - Must handle possibility of multiple roots in higher order equations
  - Variation of parameters is more complex

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Systems of Equations

- Arise in engineering problems with multiple components that are coupled
- Can convert higher order equations to system of equations
  - Sometimes reveals ideas about physical system, e.g. write equation for dy/dt and dv/dt where v = dy/dt
  - Used in numerical solutions of higher order ODEs

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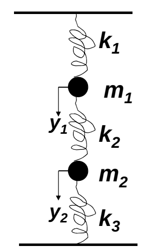
Physical Problem

- Example is system of two masses joined by a spring
  - Each mass is connected by a spring to a wall or another mass
  - Have a y coordinate for each mass
  - Both y values are zero at static equilibrium
  - F = ma for each mass



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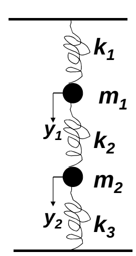
Deriving Equations

- Force on first mass is $-k_1y_1$ plus $k_2(y_2 - y_1)$
- Force on second mass is $-k_3y_2$ plus $k_2(y_1 - y_2)$
- $m_1 d^2y_1/dt^2 = -(k_1 + k_2)\,y_1 - k_2\,y_2$
- $m_2 d^2y_2/dt^2 = k_2y_1 - (k_2 + k_3)\,y_2$
- Solve both equations simultaneously



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Simultaneous Solution

$$\frac{d^2y_1}{dt^2} + \frac{k_1+k_2}{m_1}y_1 - \frac{k_2}{m_1}y_2 = 0 \qquad \frac{d^2y_2}{dt^2} - \frac{k_2}{m_2}y_1 + \frac{k_3+k_2}{m_2}y_2 = 0$$

- If we knew y_1, we could find y_2 from the first equation by algebra

$$y_2 = \frac{m_1}{k_2}\left[\frac{d^2y_1}{dt^2} + \frac{k_1+k_2}{m_1}y_1\right]$$

- Convert system into a single equation for y_1
- Start with second derivative of first equation

$$\frac{d^4y_1}{dt^4} + \frac{k_1+k_2}{m_1}\frac{d^2y_1}{dt^2} - \frac{k_2}{m_1}\frac{d^2y_2}{dt^2} = \frac{d^4y_1}{dt^4} + \frac{k_1+k_2}{m_1}\frac{d^2y_1}{dt^2} - \frac{k_2}{m_1}\left[\frac{k_2}{m_2}y_1 - \frac{k_2+k_3}{m_2}y_2\right]$$

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Simultaneous Solution II

$$\frac{d^4y_1}{dt^4}+\frac{k_1+k_2}{m_1}\frac{d^2y_1}{dt^2}-\frac{k_2}{m_1}\frac{k_2}{m_2}y_1$$
$$+\frac{k_2}{m_1}\frac{k_2+k_3}{m_2}\frac{m_1}{k_2}\left[\frac{d^2y_1}{dt^2}+\frac{k_1+k_2}{m_1}y_1\right]=0$$

- Now have one equation with only y_1
- Equation has the general form shown below

$$\frac{d^4y_1}{dt^4}+a\frac{d^2y_1}{dt^2}+by_1=0$$

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Simultaneous Solution III

- Find a and b from original equation $\frac{d^4y_1}{dt^4}+a\frac{d^2y_1}{dt^2}+by_1=0$

$$a=\frac{k_1+k_2}{m_1}+\frac{k_2}{m_1}\frac{k_2+k_3}{m_2}\frac{m_1}{k_2}=\frac{k_1+k_2}{m_1}+\frac{k_2+k_3}{m_2}$$

$$b=-\frac{k_2}{m_1}\frac{k_2}{m_2}+\frac{k_2}{m_1}\frac{k_2+k_3}{m_2}\frac{m_1}{k_2}\frac{k_1+k_2}{m_1}=\frac{k_2+k_3}{m_2}\frac{k_1+k_2}{m_1}-\frac{k_2}{m_1}\frac{k_2}{m_2}$$

- For all k and m the same $\quad a=4\frac{k}{m} \qquad b=3\left(\frac{k}{m}\right)^2$

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Simultaneous Solution IV

- For this ODE $\frac{d^4y_1}{dt^4}+a\frac{d^2y_1}{dt^2}+by_1=0$
- The solution is $y=\sum_{k=1}^{4}C_ke^{\lambda_k x}$
- Where the λ_k are solutions to $\lambda^4 + a\lambda^2 + b = 0$; solve $\mu^2 + a\mu + b = 0$ for $\mu = \lambda^2$
- Have four solutions to the equation $\lambda=\pm\sqrt{\mu}=\pm\sqrt{\frac{-a\pm\sqrt{a^2-4b}}{2}}$

$$\lambda^4+\left(\frac{k_1+k_2}{m_1}+\frac{k_3+k_2}{m_2}\right)\lambda^2+\frac{k_1+k_2}{m_1}\frac{k_3+k_2}{m_2}-\frac{k_2}{m_2}\frac{k_2}{m_1}=0$$

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Simultaneous Solution IV

- For all k and m the same: $a=4\frac{k}{m} \quad b=3\left(\frac{k}{m}\right)^2$

$$\lambda=\pm\sqrt{\mu}=\pm\sqrt{\frac{-\frac{4k}{m}\pm\sqrt{\left(\frac{4k}{m}\right)^2-(4)(3)\left(\frac{k}{m}\right)^2}}{2}}$$
$$=\pm\sqrt{-\frac{2k}{m}\pm\frac{k}{2m}\sqrt{16-12}}=\pm\sqrt{-\frac{2k}{m}\pm\frac{k}{m}}=\pm\sqrt{-\frac{3k}{m}}\ and\ \pm\sqrt{-\frac{k}{m}}$$

- Pure imaginary number gives general solution: $y_1 = A \sin \omega_1 t + B \cos \omega_1 t + C\sin \omega_2 t + D \cos \omega_2 t$ with $\omega_1^2 = k/m$ $\omega_2^2 = 3\ k/m$

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Simultaneous Solution IV

- Get y_2 from $y_2=\frac{m_1}{k_2}\left[\frac{d^2y_1}{dt^2}+\frac{k_1+k_2}{m_1}y_1\right]$
- $d^2y_1/dt^2 = -\omega_1^2 A \sin \omega_1 t - \omega_1^2 B \cos \omega_1 t - \omega_2^2 C\sin \omega_2 t - \omega_2^2 \cos \omega_2 t$
- For all k and m the same, $\omega_1^2 = k/m$ and $\omega_2^2 = 3\ k/m$

$$y_2=\frac{m_1}{k_2}\left[-\omega_1^2(A\sin\omega_1t+B\cos\omega_1t)-\omega_2^2(C\sin\omega_2t+D\cos\omega_2t)\right.$$
$$\left.+\frac{k_1+k_2}{m_1}(A\sin\omega_1t+B\cos\omega_1t+C\sin\omega_2t+D\cos\omega_2t)\right]$$

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Simultaneous Solution V

$$y_2=\frac{m_1}{k_2}\left[-\omega_1^2(A\sin\omega_1t+B\cos\omega_1t)-\omega_2^2(C\sin\omega_2t+D\cos\omega_2t)\right.$$
$$\left.+\frac{k_1+k_2}{m_1}(A\sin\omega_1t+B\cos\omega_1t+C\sin\omega_2t+D\cos\omega_2t)\right]$$

- For all k and m are the same, $\omega_1^2 = k/m$ and $m_1/k_2 = m/k$, and $(k_1 + k_2)/m_1 = 2k/m$

$$\frac{m}{k}\left[\left(-\frac{k}{m}\right)+\left(2\frac{k}{m}\right)\right]A\sin\omega_1t=A\sin\omega_1t$$
$$\frac{m}{k}\left[\left(-\frac{k}{m}\right)+\left(2\frac{k}{m}\right)\right]B\cos\omega_1t=B\cos\omega_1t$$

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Simultaneous Solution VI

$$y_2 = \frac{m_1}{k_2}\left[-\omega_1^2\left(A\sin\omega_1 t + B\cos\omega_1 t\right) - \omega_2^2\left(C\sin\omega_2 t + D\cos\omega_2 t\right) + \frac{k_1+k_2}{m_1}\left(A\sin\omega_1 t + B\cos\omega_1 t + C\sin\omega_2 t + D\cos\omega_2 t\right)\right]$$

- For all k and m are the same, $\omega_2^2 = 3k/m$ and $m_1/k_2 = m/k$, and $(k_1 + k_2)/m_1 = 2k/m$

$$\frac{m}{k}\left[\left(-\frac{3k}{m}\right) + \left(2\frac{k}{m}\right)\right] C\sin\omega_2 t = -C\sin\omega_2 t$$

$$\frac{m}{k}\left[\left(-\frac{3k}{m}\right) + \left(2\frac{k}{m}\right)\right] D\cos\omega_2 t = -D\cos\omega_2 t$$

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Simultaneous Solution VII

$$y_2 = A\sin\omega_1 t + B\cos\omega_1 t - C\sin\omega_2 t - D\cos\omega_2 t$$

$$y_1 = A\sin\omega_1 t + B\cos\omega_1 t + C\sin\omega_2 t + D\cos\omega_2 t$$

- Determine A, B, C, and D from initial conditions on the two mass positions and velocities, $y_1(0)$, $y_1'(0)$, $y_2(0)$, and $y_2'(0)$

$$y_1(0) = B + D \qquad y_2(0) = B - D$$

$$y_1'(0) = \omega_1 A + \omega_2 C \qquad y_2'(0) = \omega_1 A - \omega_2 C$$

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Simultaneous Solution VIII

$$y_1(0) = B + D \qquad y_2(0) = B - D$$

$$y_1'(0) = \omega_1 A + \omega_2 C \qquad y_2'(0) = \omega_1 A - \omega_2 C$$

$$B = \frac{y_1(0) + y_2(0)}{2} \qquad D = \frac{y_1(0) - y_2(0)}{2}$$

$$A = \frac{y_1'(0) + y_2'(0)}{2\omega_1} \qquad C = \frac{y_1'(0) - y_2'(0)}{2\omega_2}$$

$$y_1 = A\sin\omega_1 t + B\cos\omega_1 t + C\sin\omega_2 t + D\cos\omega_2 t$$

$$y_2 = A\sin\omega_1 t + B\cos\omega_1 t - C\sin\omega_2 t - D\cos\omega_2 t$$

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Simultaneous Solution IX

- Initial condition of zero velocity and equal displacement towards center
- $y_1(0) = a$, $y_2(0) = -a$, $y_1'(0) = y_2'(0) = 0$

$$B = \frac{y_1(0) + y_2(0)}{2} = \frac{a - a}{2} = 0 \qquad D = \frac{y_1(0) - y_2(0)}{2} = \frac{a - (-a)}{2} = a$$

$$A = \frac{y_1'(0) + y_2'(0)}{2\omega_1} = 0 \qquad C = \frac{y_1'(0) - y_2'(0)}{2\omega_2} = 0$$

$$y_1 = a\cos\omega_2 t \qquad y_2 = -a\cos\omega_2 t \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

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Simultaneous Summary

- Differentiate first (y_1) equation to obtain derivative(s) of y_2 that are present in second equation
- Solve first equation for y_2
- Substitute equations for y_2 and its derivatives into the result of the first step
- Result is a higher order equation that can be solved (if possible) by usual methods

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Matrix Differential Equations

- Consider a system of n variables $y_i(t)$
- Each variable is described by a differential equation that has a linear dependence on all the other y_k
- Show equations as individual equations and in matrix form

$$\frac{dy_i}{dt} + \sum_{j=1}^{n} a_{ij} y_j = r_i(t) \qquad i = 1, \ldots n$$

$$\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = \mathbf{r}$$

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Matrix Differential Equations II

- Matrix components in $\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = \mathbf{r}$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} \quad \frac{d\mathbf{y}}{dt} = \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} dy_1/dt \\ dy_2/dt \\ dy_3/dt \\ \vdots \\ \vdots \\ dy_n/dt \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{r} = [r_1 \quad r_2 \quad r_3 \quad \cdots \quad \cdots \quad r_n]^T$$

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Solving $\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = \mathbf{r}$

- Assume that $\mathbf{A}_{(n \times n)}$ has n linearly independent eigenvectors
- Eigenvectors are columns of matrix, **X**
- Define a new vector **s** = $\mathbf{X}^{-1}\mathbf{y}$ (**y** = **Xs**)
- Substitute **y** = **Xs** into the matrix differential equation
- Since **A** components are constants, **X** components are constants

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Solving $\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = \mathbf{r}$ Continued

$$\mathbf{r} = \frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = \frac{d\mathbf{X}\mathbf{s}}{dt} + \mathbf{A}\mathbf{X}\mathbf{s} = \mathbf{X}\frac{d\mathbf{s}}{dt} + \mathbf{A}\mathbf{X}\mathbf{s} = \mathbf{r}$$

- Premultiply last equation by $\mathbf{X}^{-1}$
- $\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \Lambda$, a diagonal matrix whose components are **A** eigenvalues

$$\mathbf{X}^{-1}\mathbf{X}\frac{d\mathbf{s}}{dt} + \mathbf{X}^{-1}\mathbf{A}\mathbf{X}\mathbf{s} = \mathbf{X}^{-1}\mathbf{r} \quad \Rightarrow \quad \frac{d\mathbf{s}}{dt} + \Lambda\mathbf{s} = \mathbf{X}^{-1}\mathbf{r} = \mathbf{p}$$

- Each **p** component may contain all the r_i

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Structure of $\frac{d\mathbf{s}}{dt} + \Lambda\mathbf{s} = \mathbf{p}$

$$\begin{bmatrix} ds_1/dt \\ ds_2/dt \\ ds_3/dt \\ \vdots \\ \vdots \\ ds_n/dt \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ p_n \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ p_n \end{bmatrix} - \begin{bmatrix} \lambda_1 s_1 \\ \lambda_2 s_2 \\ \lambda_3 s_3 \\ \vdots \\ \vdots \\ \lambda_n s_n \end{bmatrix}$$

- Result is set of n scalar differential equations: $ds_i/dt = p_i - \lambda_i s_i$

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Solution of $\frac{d\mathbf{s}}{dt} + \Lambda\mathbf{s} = \mathbf{p}$

- We know the solution to $ds_i/dt + \lambda_i s_i = p_i$, (general linear first order ODE) $\quad s_i = e^{-\lambda_i t}\left[\int e^{\lambda_i t} p_i dt + C_i\right]$
- Define q_i as integral $\quad q_i = e^{-\lambda_i t}\int e^{\lambda_i t} p_i dt$
- Solution for **s** components $\quad s_i = C_i e^{-\lambda_i t} + q_i$

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Matrix Solution Terms $s_i = C_i e^{-\lambda_i t} + q_i$

$$\mathbf{E}(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & e^{-\lambda_3 t} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & e^{-\lambda_n t} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_n \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ \vdots \\ q_n \end{bmatrix}$$

- With these definitions, $s_i = C_i e^{\lambda_i t} + q_i$ becomes **s** = **E**(t)**C** + **q** (at t = 0, **E**(0) = **I**)

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Apply Initial Conditions on **y**(0)

- Get **y** = **Xs** = **XEC** + **Xq**
- At t = 0, **E** = **I**, and **y** = $\mathbf{y}_0$ = initial **y** components, giving $\mathbf{y}_0 = \mathbf{XIC} + \mathbf{Xq}_0$
- Premultiply by $\mathbf{X}^{-1}$ to obtain $\mathbf{X}^{-1}\mathbf{y}_0 = \mathbf{X}^{-1}\mathbf{XC} + \mathbf{X}^{-1}\mathbf{Xq}_0 = \mathbf{C} + \mathbf{q}_0$
- Constant vector, $\mathbf{C} = \mathbf{X}^{-1}\mathbf{y}_0 - \mathbf{q}_0$
- Result: $\mathbf{y} = \mathbf{XE}\,[\mathbf{X}^{-1}\mathbf{y}_0 - \mathbf{q}_0] + \mathbf{Xq}$
- Homogenous (**q** = **0**): $\mathbf{y} = \mathbf{XEX}^{-1}\mathbf{y}_0$

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Complex Roots

- Will produce solutions in complex conjugate pairs
- Exponential solutions can then be decomposed into real and imaginary parts
- Imaginary exponentials form sine and cosine solutions

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Reduction of Order Example

- An n^{th} order equation can be written as a system of n first order equations
- $d^3y/dx^3 + g\, d^2y/dx^2 + h\, dy/dx + f\, y = k$
- Define $z = dy/dx$, $w = dz/dx = d^2y/dx^2$, which gives $dw/dx = d^3y/dx^3$
- We now have three first-order ODEs
  – $dw/dx = k - g\,w - h\,z - f\,y$
  – $dz/dx = w$
  – $dy/dx = z$

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Reduction of Order

- An n^{th} order equation can be written as a system of n first order equations
- Consider a general nonlinear n^{th} order equation as shown below

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

- Define a sequence of variables, z_k
  – Start with $z_1 = y$, $z_2 = dz_1/dx = dy/dx$, and continuing as $z_k = dz_{k-1}/dx = d^{k-1}y/dx^{k-1}$

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Reduction of Order II

- Continue this definition up to z_n, where $dz_n/dx = d^ny/dx^n$

$$z_{n-1} = \frac{dz_{n-2}}{dx} = \frac{d^{n-2}y}{dx^{n-2}} \qquad z_n = \frac{dz_{n-1}}{dx} = \frac{d^{n-1}y}{dx^{n-1}}$$

- z_1 to z_n are n variables that satisfy simultaneous, first-order ODEs

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{dz_n}{dx} = f(x, z_1, z_2, \cdots, z_n)$$

$$\frac{dz_k}{dx} = z_{k+1} \qquad k = 1, \ldots, n-1$$

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Reduction of Order III

- The main application of reduction of order is to numerical methods
- Numerical solutions of ODEs develop methods to solve a system of first order equations
- Higher order equations are solved by converting them to a system of first order equations

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Reduction of Order Example

$$\frac{d^2y_1}{dt^2}+\frac{k_1+k_2}{m_1}y_1-\frac{k_2}{m_1}y_2=0 \qquad \frac{d^2y_2}{dt^2}-\frac{k_2}{m_2}y_1+\frac{k_3+k_2}{m_2}y_2=0$$

- Define variables $y_3 = dy_1/dt$ and $y_4 = dy_2/dt$
- Then $dy_3/dt = d^2y_1/dt^2$ and $dy_4/dt = d^2y_2/dt^2$
- Have four simultaneous first-order ODEs

$$\frac{dy_1}{dt}-y_3=0 \qquad \frac{dy_2}{dt}-y_4=0$$
$$\frac{dy_3}{dt}+\frac{k_1+k_2}{m_1}y_1-\frac{k_2}{m_1}y_2=0$$
$$\frac{dy_4}{dt}-\frac{k_2}{m_2}y_1+\frac{k_3+k_2}{m_2}y_2=0$$

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Write as Matrix Equation

- Write four equations as $\frac{d\mathbf{y}}{dt}+\mathbf{Ay}=\mathbf{0}$

$$\begin{bmatrix} dy_1/dt \\ dy_2/dt \\ dy_3/dt \\ dy_4/dt \end{bmatrix}+\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 & 0 \\ -\frac{k_2}{m_2} & \frac{k_3+k_2}{m_2} & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Solve the Matrix Equation

- From array solution, we have solution to $d\mathbf{y}/dx + \mathbf{Ay} = \mathbf{0}$ as $\mathbf{y} = \mathbf{XEX}^{-1}\mathbf{y}_0$
  - $\mathbf{X}$ is eigenvector matrix, $\mathbf{y}_0$ is initial condition vector and $\mathbf{E}$ is shown below

$$\mathbf{E}(t)=\begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & e^{-\lambda_3 t} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & e^{-\lambda_n t} \end{bmatrix}$$

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Get Eigenvalues

$$Det(\mathbf{A}-\mathbf{I}\lambda)=\begin{vmatrix} -\lambda & 0 & -1 & 0 \\ 0 & -\lambda & 0 & -1 \\ \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & -\lambda & 0 \\ -\frac{k_2}{m_2} & \frac{k_3+k_2}{m_2} & 0 & -\lambda \end{vmatrix}$$

- Expand determinant along first row

$$=-\lambda\begin{vmatrix} -\lambda & 0 & -1 \\ -\frac{k_2}{m_1} & -\lambda & 0 \\ \frac{k_3+k_2}{m_2} & 0 & -\lambda \end{vmatrix}-1\begin{vmatrix} 0 & -\lambda & -1 \\ \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_3+k_2}{m_2} & -\lambda \end{vmatrix}$$

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Get Eigenvalues II

$$Det(\mathbf{A}-\mathbf{I}\lambda)=-\lambda\begin{vmatrix} -\lambda & 0 & -1 \\ -\frac{k_2}{m_1} & -\lambda & 0 \\ \frac{k_3+k_2}{m_2} & 0 & -\lambda \end{vmatrix}-1\begin{vmatrix} 0 & -\lambda & -1 \\ \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_3+k_2}{m_2} & -\lambda \end{vmatrix}$$

$$=(-\lambda)\left[(-\lambda)^3+0+0+\frac{k_3+k_2}{m_2}(-\lambda)-0-0\right]-1\left[0-\frac{k_1+k_2}{m_1}\frac{k_3+k_2}{m_2}+0-\frac{k_2}{m_2}\frac{k_2}{m_1}\right.$$
$$\left.-(-\lambda)^2\frac{k_1+k_2}{m_1}\right]=\lambda^4+\left(\frac{k_1+k_2}{m_1}+\frac{k_3+k_2}{m_2}\right)\lambda^2+\frac{k_1+k_2}{m_1}\frac{k_3+k_2}{m_2}-\frac{k_2}{m_2}\frac{k_2}{m_1}=0$$

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Example Concluded

- We see that eigenvalue equation is the same as the one obtained by simultaneous solution
- To proceed with this approach we would have to find eigenvectors to get **X** matrix and its inverse
- This approach is best when we have a symmetric **A** matrix (so that $\mathbf{X}^{-1} = \mathbf{X}^T$) and eigenvalues are real

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Supplemental Material

- The following charts are are intended to show comparison of class material
- The first set of charts shows the solution of the equation d**y**/dx + **Ay** = **r** for constant **r**
- The second set of charts shows the equivalence of the second order differential equation solution and the eigenvalue solution for λ

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$\frac{d\mathbf{s}}{dt} + \Lambda\mathbf{s} = \mathbf{p}$ solution with constant **p**

- Solution from previous chart $s_i = e^{-\lambda_i t}\left[\int e^{\lambda_i t} p_i dt + C_i\right]$
- If p_i is constant (or zero) we have $s_i = C_i e^{-\lambda_i t} + p_i/\lambda_i$
- How do we write p_i/λ_i in matrix notation?
- Start with $\Lambda^{-1}\mathbf{p}$ where Λ^{-1} is diagonal matrix whose components are $1/\lambda_i$
  - For **A** = diag(a_i), $\mathbf{A}^{-1}$ = **B** = diag($1/a_i$)

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Terms in Solution of $\frac{d\mathbf{s}}{dt} + \Lambda\mathbf{s} = \mathbf{p}$

$$\mathbf{E}(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & e^{-\lambda_3 t} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & e^{-\lambda_n t} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_n \end{bmatrix} \quad \Lambda^{-1}\mathbf{p} = \begin{bmatrix} p_1/\lambda_1 \\ p_2/\lambda_2 \\ p_3/\lambda_3 \\ \vdots \\ \vdots \\ p_n/\lambda_n \end{bmatrix}$$

- With these definitions, $s_i = C_i e^{\lambda_i t} + p_i/\lambda_i$ becomes **s** = **EC** + Λ^{-1}**p** (**E**(0) = **I**)

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Apply Initial Conditions on **y**(0)

- Get **y** = **Xs** = **XEC** + **X**Λ^{-1}**p**
- At t = 0, **E** = **I**, and **y** = $\mathbf{y}_0$ = initial **y** components, giving $\mathbf{y}_0$ = **XIC** + **X**Λ^{-1}**p**
- Premultiply by $\mathbf{X}^{-1}$ to obtain $\mathbf{X}^{-1}\mathbf{y}_0 = \mathbf{X}^{-1}\mathbf{XC} + \mathbf{X}^{-1}\mathbf{X}\Lambda^{-1}\mathbf{p} = \mathbf{C} + \Lambda^{-1}\mathbf{p}$
- Constant vector, $\mathbf{C} = \mathbf{X}^{-1}\mathbf{y}_0 - \Lambda^{-1}\mathbf{p}$
- Result: $\mathbf{y} = \mathbf{XE}\,[\mathbf{X}^{-1}\mathbf{y}_0 - \Lambda^{-1}\mathbf{p}] + \mathbf{X}\Lambda^{-1}\mathbf{p}$
- Homogenous(**p** = **0**): $\mathbf{y} = \mathbf{XEX}^{-1}\mathbf{y}_0$

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Second Order Example

- Show connection between eigenvalue equation for λ and ODE equation for λ

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0 \qquad Let\ y_1 = x\ and\ y_2 = \frac{dx}{dt}$$

$$\frac{d\mathbf{y}}{dt} + \mathbf{Ay} = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ b & a \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

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Second Order Example

- Get eigenvalues of **A** matrix

$$Det(\mathbf{A} - \mathbf{I}\lambda) = \begin{vmatrix} -\lambda & -1 \\ b & a-\lambda \end{vmatrix} = -\lambda(a-\lambda) + b = 0$$

$$(-\lambda)^2 + a(-\lambda) + b = 0$$

- Same equation as for second order equation except for sign change
  - Sign change necessary for use in formulas (λ in second order equation is –λ for array.)

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